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Mean field theory for usp(4) ≃ so(5)

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Abstract

The simple Lie algebra usp(4) ≃ so(5) is a ten-dimensional algebra that contains the angular momentum algebra su(2). In nuclear structure physics, the algebra so(5) describes beta-rigid collective modes in the Bohr–Mottelson and interacting boson models. The so(5) dual space consists of density matrices which are defined by the expectations of so(5) generators. A coadjoint orbit is a common level surface in the dual space of the two so(5) Casimirs. This paper develops mean field theory on any coadjoint orbit of so(5) densities. When the densities of a set of quantum states lie on one orbit, the system is said to have a weak dynamical symmetry. A Lax pair determines the dynamics of so(5) densities on each coadjoint orbit. Analytic solutions are reported for rotating so(5) densities in equilibrium for a particular energy function.

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1. Introduction

Algebraic models of many-body quantum mechanical systems usually postulate that a subset of observed states of physical interest are vectors in one irreducible representation (irrep) of some Lie algebra. When this postulate is not satisfied, a kind of algebraic model, called an algebraic mean field theory, can still be constructed and applied advantageously to some interesting physical systems. For a given Lie algebra and group, the mathematical setting for mean field theory is a coadjoint orbit of the Lie group in the dual space to its Lie algebra. This paper applies the mean field method to the so(5) ≅ usp(4) algebra which is relevant to physical applications in nuclear structure [1, 2] and high-temperature superconductivity [3–5]. In prior work the algebraic mean field method was applied in the field of nuclear structure physics to the Elliott su(3) model [6–10], the symplectic sp(3,R) collective model [11–14], and the gcm(3) general collective motion and Riemann ellipsoid model [15]. The first part of this introduction motivates the mean field method, and provides some background, examples and perspective. The second part defines the matrix Lie algebra usp(4) ≅ so(5), chooses a

so(3)-adapted basis for the algebra, and assumes a (generally reducible) usp(4) $\cong$ so(5) irrep, which provides the physical interpretation for the particular application at hand.

To understand how a 'dynamical symmetry' can be present even though the single irrep ansatz fails, it is useful to consider two illustrative examples. The first example is rotational bands, as found commonly in deformed atomic nuclei [16]. The experimental signatures for such bands are as follows: (1) a sequence of states with different angular momenta L whose excitation energies are approximately proportional to the square $L(L+1)$ of the angular momentum, and (2) for which there are strong electric quadrupole transition rates among the states. The band's constant moment of inertia depends on the deformation which is measured by the quadrupole tensor $Q_\mu = \sum r^2 Y^{(2)}_\mu(\Omega)$, where the sum is over all the particles making up the system. The electric transition rates are proportional to the square of the matrix elements of the operator Q_μ. The regularity in the experimental spectrum and the strong electric transitions suggest a dynamical symmetry based on the Lie algebra rot(3), an eight-dimensional semidirect sum spanned by the angular momentum algebra so(3) and the five-dimensional Abelian ideal generated by the quadrupole tensor [17, 18].

The algebra rot(3) includes sufficient degrees of freedom to model rotating systems of particles. The Mackey inducing construction determines all rot(3) irreps [18]. With respect to energy levels and transition rates, the rot(3) irreps are indistinguishable from the Bohr–Mottelson geometrical model of the nucleus as a rotating droplet with a fixed quadrupole deformation [19]. Rotational bands with their attendant experimental signatures are described rather accurately in many cases by one rot(3) irrep. The evidence suggests the rot(3) algebra determines a dynamical symmetry for nuclear rotational bands. On closer inspection, however, there is a significant mathematical obstruction. Within each rot(3) irrep, the operators representing Casimir elements are a multiple of the identity. There are two rot(3) Casimirs given by angular momentum coupling of the quadrupole tensor to zero angular momentum, $[Q \times Q]^{(0)}$ and $[Q \times Q \times Q]^{(0)}$. These two Casimirs measure the intrinsic quadrupole deformation of the rotating system [20, 21]. Because each Casimir is a constant in a rot(3) irrep, the deformation and, hence, the moment of inertia are constant too. Yet on the Fock space of many particles, these Casimirs are pure multiplication operators and have no eigenstates. Indeed the decomposition of rot(3) on Fock space requires a direct integral and the mathematics does not allow for any Fock space state to be a vector in a single rot(3) irrep. In physical terms, any rot(3) irrep vector has zero quantum fluctuations in its intrinsic quadrupole deformation, a circumstance that violates the Heisenberg uncertainty principle and cannot be realized in nature.

The wavefunctions in the Fock space for real rotational bands must be direct integrals of a continuous range of rot(3) irrep states which are labelled by the Casimirs or, equivalently, the deformation. To yield the observed constant moment of inertia for a particular band, the support for the integrals must be concentrated in a narrow zone of deformations close to their experimental values. The expectation values for any operator in the rot(3) algebra do not depend sensitively on the details of the integral, provided its support is concentrated. However, the expectations for operators that are not in the rot(3) algebra do depend on such details. For the purpose of modelling rotational bands, we are most interested in operators in the rot(3) algebra and, thus, the details of the wavefunctions are not a major concern. To enjoy an economical theory of rotational bands and, thereby, respect Ockham's razor, this analysis strongly suggests changing the mathematical framework, if feasible, from wavefunctions to expectations of algebra operators.

The fundamental mathematical object of algebraic mean field theory is the density matrix which is defined by the quantum expectations of the operators in the Lie algebra. The space of all densities is the dual space to the Lie algebra. The Casimirs are smooth real-valued

functions of the density. For rot(3) rotational bands, a constant moment of inertia requires that the rot(3)-densities of the states forming the band must share the same values for the two Casimir functions. Therefore the densities of the band's states must lie on a common level surface of the rot(3) Casimirs. Although nuclear rotational bands cannot satisfy the dynamical symmetry postulate of a single rot(3) irrep, the much weaker hypothesis that the densities are points on a level surface of the rot(3) Casimirs is confirmed by experimental measurements of constant intrinsic nuclear quadrupole deformation [22].

A nuclear rotational band is said to have a weak rot(3) dynamical symmetry in the following sense.

Definition. *A subset of quantum states has a weak dynamical symmetry relative to some Lie algebra* g *when the* g*-densities lie on one level surface of the algebra's Casimir functions.*

When the subset of states has a dynamical symmetry relative to g in the usual sense, these states are vectors from one irrep of g, and the corresponding densities are points on one integral coadjoint orbit. Since the Casimir functions are constant on each coadjoint orbit, a dynamical symmetry in the usual sense is necessarily a weak dynamical symmetry. But the converse is not true.

A second, more topical, example of the failure of the single irrep postulate arises in the description of quantum phase transitions in the interacting boson model (IBM) [23–25]. Consider two subalgebras, su(3) and u(5), of the IBM algebra u(6), and two Hamiltonians, H_{su3} and H_{u5}, defined by two Casimir elements of su(3) and u(5), respectively. Define the Hamiltonian H_α as the interpolation between these two Casimir Hamiltonians, $H_\alpha = \alpha H_{\mathrm{su3}} + (1-\alpha)H_{\mathrm{u5}}$, where α is a real parameter, and consider its spectrum within a symmetric u(6) irrep $[N]$, where N denotes the number of bosons. Pure dynamical symmetry for H_α is attained only when $\alpha = 0$ (u(5) symmetry) or $\alpha = 1$ (su(3) symmetry). Numerical studies show, however, that there exists a critical value $\alpha_{\mathrm{cr}} \approx 0.5$ that separates a u(5) domain for $0 \leqslant \alpha < \alpha_{\mathrm{cr}}$ from a su(3) domain for $\alpha_{\mathrm{cr}} < \alpha \leqslant 1$. Within these two domains the two symmetries persist, despite the loss of dynamical symmetry in the strong sense. There is a transition zone around $\alpha = \alpha_{\mathrm{cr}}$ where neither symmetry is present; this transition zone narrows with increasing boson number N.

In particular, for $\alpha_{\mathrm{cr}} < \alpha \leqslant 1$, the low-energy spectrum of H_α shows a band structure and electric transitions that are recognizable signatures of the su(3) dynamical symmetry [26, 27]. These signatures suggest that, at least approximately, the su(3) densities of the H_α low-energy eigenstates for $\alpha > \alpha_{\mathrm{cr}}$ lie on one level surface of the su(3) Casimirs. A close inspection of the eigenstates demonstrates weak su(3) dynamical symmetry for α above the critical point. Figure 1 is a histogram showing the su(3) amplitudes for the least energy eigenstates of H_α with angular momenta $L = 0, 2, 4$ just above the critical point for the u(6) irrep with $N = 100$ bosons. The amplitudes are distributed over dozens of inequivalent su(3) irreps, but only the largest amplitudes are shown in the figure. The key observation is that the amplitudes are only weakly dependent on the angular momentum. For larger α the L-independence of the amplitudes is even more pronounced until, when $\alpha = 1$, all amplitudes vanish except for one. The low-energy eigenstates ψ_{LM} are, according to the numerical calculations, given by

$$\psi_{LM} = \sum_{(\lambda\mu)} A(\lambda\mu)|(\lambda\mu)LM\rangle, \tag{1}$$

where $(\lambda\mu)$ labels the su(3) irreps in the basis of the u(6) irrep $N = 100$ and $A(\lambda\mu)$ denotes the L, M independent amplitude with respect to the su(3) adapted basis state $|(\lambda\mu)LM\rangle$. If $\hat{C}$ denotes the representation of any su(3) Casimir element and $C(\lambda\mu)$ its value within a su(3)

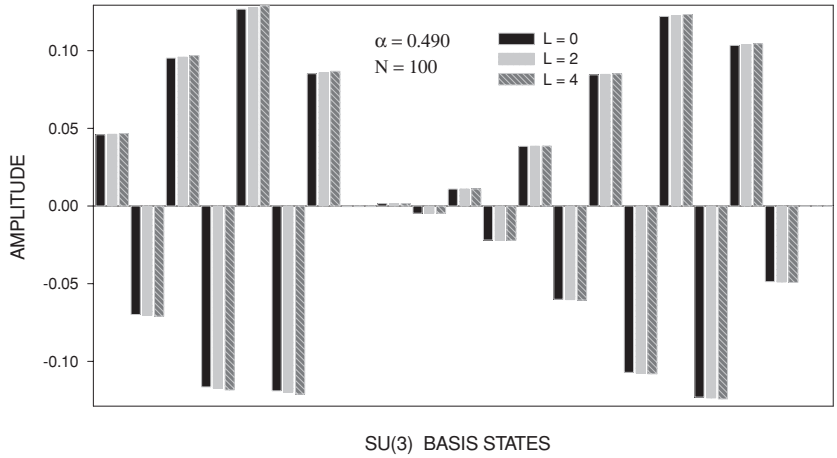

Figure 1. Amplitudes of the least energy eigenstates of H_α with respect to a su(3) basis for angular momenta $L = 0$, 2, 4 and for the u(6) irrep with $N = 100$ bosons. This u(6) irrep contains 884 su(3) irreps, but only 21 of them with the largest amplitudes are shown in the bar graph. The included (λ, μ) su(3) irreps are, from left to right, as follows: (116,30), (118,26), (120,22), (122,18), (124,14), (126,10), (128,6), (130,2), (104,48), (106,44), (108,40), (110,36), (112,32), (114,28), (116,24), (118,20), (120,16), (122,12), (124,8), (126,4), (128,0).

irrep, then its expectation with respect to an eigenstate ψ_{LM} is independent of the angular momentum:

$$\langle \psi_{LM} | \hat{C} | \psi_{LM} \rangle = \sum_{(\lambda\mu)} |A(\lambda\mu)|^2 C(\lambda\mu). \tag{2}$$

Therefore, the densities of the eigenstates lie on a level surface of the Casimirs, and weak su(3) dynamical symmetry is proven for the set of states ψ_{LM}.

The complex interplay of many physical degrees of freedom determines the absolute energies and wavefunctions of quantum many-body systems. Except for special cases, no dynamical symmetry algebra exists to shed light on the intricate structure of quantum states. But, as the examples illustrate, the relationships, namely, excitation energies and electric transitions, among the ground and some selected low-energy excited states can be rather simple—sufficiently so that the system admits a weak dynamical symmetry.

Although the two chosen examples are drawn from nuclear structure physics, the weak dynamical symmetry ansatz is not specific to atomic nuclei and can be applied to other quantum many-body systems. Hartree–Fock and Hartree–Fock–Bogoliubov, the exemplars of mean field theory, are instances of weak dynamical symmetry relative to unitary and orthogonal algebras, respectively [28–30]. The name 'algebraic mean field theory' alludes to these paradigm theories, and connotes the generalization to any Lie algebra.

In this paper, the algebraic mean field method is applied to the ten-dimensional simple Lie algebra usp(4) $\cong$ so(5). The primary goal is to derive analytic results for the energies and densities of excited states from the weak dynamical symmetry ansatz. This goal is achieved in section 3.4. A secondary goal is to develop tractable methods that can be applied successfully to the mean field theory of other Lie algebras. The mean field theory for the usp(4) algebra requires different, somewhat less elegant, techniques than prior investigations for su(3) and sp(3,R) because the rotation group SU(2) is not embedded canonically in USp(4).

In the next subsection, the paper defines the matrix Lie algebra usp(4) and assumes a unitary representation π of it is given. The representation determines the physical interpretation. It might be associated with the o(5) limits of the interacting boson model [1] and the related geometrical collective model [2]. Other representations of this algebra occur in nuclear physics in connection with neutron–proton pairing [31, 32] and neutrino science [33]. The algebra o(5) also finds applications to high-temperature superconductivity [3–5]. The mathematical theory developed in this paper may be applied to any of these physical models with appropriate modifications depending on the physical interpretation.

To set up the mean field theory of the matrix algebra usp(4), this paper carries out a sequence of well-defined steps. Section 2 determines explicitly the dual space usp(4)*, consisting of all usp(4) density matrices, and the coadjoint action of the Lie group USp(4) on the dual space. The faithful matrix representation of usp(4) ≃ so(5) as 4×4 complex matrices is used for this purpose because it is the one with the smallest dimension. In the mean field approximation, the model densities are limited to one coadjoint orbit. The set of all such orbits is enumerated next. The usp(4) Casimirs are constant real-valued functions on each coadjoint orbit. In fact a level surface of the Casimirs is a coadjoint orbit; the characterization of a coadjoint orbit as a level surface is useful for mean field computations. Among the level surfaces are the integral coadjoint orbits associated with the coherent states of the highest weight irreducible representations of usp(4) ≃ so(5) [34].

Section 3 defines the symplectic geometry of a coadjoint orbit [35]. This geometry associates a Hamiltonian vector field to each smooth function on a coadjoint orbit. In particular, the mean field Hamiltonian is the vector field corresponding to the energy function. Such vector fields are usp(4)-valued functions of the density. The Hamiltonian vector fields associated with several functions of physical interest are determined. The equation of motion for densities is a Hamiltonian dynamical system on a coadjoint orbit. The dynamics is a finite-dimensional Lax system [36, 37].

When the energy function is rotationally invariant, the dynamics on a coadjoint orbit simplifies to a Lax system on a proper submanifold of the orbit space. On this submanifold, the angular momentum vector is aligned with the 3-axis. Section 4 reports analytic solutions for a particular energy function.

1.1. Definition of usp(4)

Suppose $\hat{L}^{(1)}_\mu$, $\mu = 0, \pm 1$ denote the spherical components of an angular momentum tensor operator, spanning the Lie algebra su(2), and $\hat{O}^{(3)}_\nu$, $\nu = 0, \pm 1, \pm 2, \pm 3$, denote the components of an octupole tensor operator. These (dimensionless) operators are assumed to be Hermitian when acting on a Hilbert space $\mathcal{H}$ and to obey the commutation relations,

$$\begin{aligned} [\hat{O}^{(3)} \times \hat{O}^{(3)}]^{(1)} &= -\sqrt{7}\hat{L}^{(1)} \\ [\hat{O}^{(3)} \times \hat{O}^{(3)}]^{(3)} &= -\frac{\sqrt{6}}{2}\hat{O}^{(3)} \\ [\hat{O}^{(3)} \times \hat{O}^{(3)}]^{(5)} &= 0. \end{aligned} \tag{3}$$

The algebra of operators $\{\hat{L}^{(1)}_\mu, \hat{O}^{(3)}_\nu\}$ does not necessarily act irreducibly on $\mathcal{H}$. For example, in either the Bohr–Mottelson collective model or the interacting boson model, such operators are given by $\hat{L}^{(1)} = \sqrt{10}[d^\dagger \times \tilde{d}]^{(1)}$ and $\hat{O}^{(3)} = -\sqrt{10}[d^\dagger \times \tilde{d}]^{(3)}$, where $d^\dagger$ and $\tilde{d}$ denote the creation and destruction tensor operators for the spherical components of a $L = 2$ d-boson.

The set of ten operators $\{\hat{L}^{(1)}_\mu, \hat{O}^{(3)}_\nu\}$ closes under commutation to form a Lie algebra that will be shown now to be a unitary representation of usp(4) ≅ so(5). The unitary symplectic matrix Lie algebra usp(4) is defined by

Table 1. Hamiltonian vector fields corresponding to elementary functions and a basis for the complexification of usp(4).

Function f	Hamiltonian vector field $S_f[\rho]$
$\lambda(S)$	S
l_0	$\mathcal{L}_0 = \frac{3}{2}E_{11} + \frac{1}{2}E_{22} - \frac{1}{2}E_{33} - \frac{3}{2}E_{44}$
l_1	$\mathcal{L}_1 = -\sqrt{\frac{3}{2}}E_{21} - \sqrt{2}E_{32} - \sqrt{\frac{3}{2}}E_{43}$
l_{-1}	$\mathcal{L}_{-1} = \sqrt{\frac{3}{2}}E_{12} + \sqrt{2}E_{23} + \sqrt{\frac{3}{2}}E_{34}$
o_0	$\mathcal{O}_0 = \frac{1}{2}E_{11} - \frac{3}{2}E_{22} + \frac{3}{2}E_{33} - \frac{1}{2}E_{44}$
o_1	$\mathcal{O}_1 = -E_{21} + \sqrt{3}E_{32} - E_{43}$
o_{-1}	$\mathcal{O}_{-1} = E_{12} - \sqrt{3}E_{23} + E_{34}$
o_2	$\mathcal{O}_2 = \sqrt{\frac{5}{2}}E_{31} - \sqrt{\frac{5}{2}}E_{42}$
o_{-2}	$\mathcal{O}_{-2} = \sqrt{\frac{5}{2}}E_{13} - \sqrt{\frac{5}{2}}E_{24}$
o_3	$\mathcal{O}_3 = -\sqrt{5}E_{41}$
o_{-3}	$\mathcal{O}_{-3} = \sqrt{5}E_{14}$

$$\mathrm{usp}(4) = \{S \in M_4(C) | {}^T S \cdot J + J \cdot S = 0, S^\dagger = S\} \tag{4}$$

$$= \left\{ S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_4(C) | a, b, c, d \in M_2(C), \right.$$
$$\left. d = K \cdot {}^T a \cdot K, {}^T b = K \cdot b \cdot K, a^\dagger = a, c = b^\dagger \right\} \tag{5}$$

where J and K are the nondegenerate antisymmetric matrices,

$$J = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \in M_4(R), \qquad K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M_2(R). \tag{6}$$

$M_n(R)$, respectively $M_n(C)$, denotes the algebra of $n \times n$ real, respectively complex, matrices. A left superscript T indicates matrix transposition and a right superscript † denotes Hermitian conjugation, $S^\dagger = {}^T S^*$. Strictly speaking, the elements of the real Lie algebra usp(4) should be skew-Hermitian matrices instead of Hermitian matrices, but the correspondence with physics is enhanced using Hermitian S. The unitary symplectic group is the connected and simply connected matrix Lie group,

$$\mathrm{USp}(4) = \{g \in M_4(C) | {}^T g \cdot J \cdot g = J, g^\dagger \cdot g = \mathrm{Id}\}, \tag{7}$$

where Id denotes the identity matrix. When S is a Lie algebra element in usp(4), its exponentiation $\exp(\mathrm{i}S)$ is a matrix in the Lie group USp(4).

A basis for the complexification of the real Lie algebra usp(4) is the set of ten matrices $\{\mathcal{L}_\mu, \mathcal{O}_\nu\}$, which are defined in the second column of table 1. E_{ij} denotes the elementary matrix all of whose entries are zero except for the entry equal to one at the intersection of row i and column j. The usp(4) basis matrices satisfy the identities, $(\mathcal{L}_\mu)^\dagger = \mathcal{L}_{-\mu}$ and $(\mathcal{O}_\nu)^\dagger = \mathcal{O}_{-\nu}$. A general element $S(u, v)$ of usp(4) is defined by a set of ten complex numbers $\{u_\mu, v_\nu\}$ satisfying $u_{-\mu} = (-1)^\mu (u_\mu)^*$ and $v_{-\nu} = (-1)^\nu (v_\nu)^*$:

$$S(u, v) = u \cdot \mathcal{L} + v \cdot \mathcal{O} = \sum_{\mu=-1}^{1} (-1)^\mu u_{-\mu} \mathcal{L}_\mu + \sum_{\nu=-3}^{3} (-1)^\nu v_{-\nu} \mathcal{O}_\nu. \tag{8}$$

When written explicitly, the general matrix in usp(4) is

$$S(u,v)=\begin{pmatrix} \frac{1}{2}v_0+\frac{3}{2}u_0 & -v_1-\sqrt{\frac{3}{2}}u_1 & \sqrt{\frac{5}{2}}v_2 & -\sqrt{5}v_3 \\ v_{-1}+\sqrt{\frac{3}{2}}u_{-1} & \frac{1}{2}u_0-\frac{3}{2}v_0 & \sqrt{3}v_1-\sqrt{2}u_1 & -\sqrt{\frac{5}{2}}v_2 \\ \sqrt{\frac{5}{2}}v_{-2} & \sqrt{2}u_{-1}-\sqrt{3}v_{-1} & \frac{3}{2}v_0-\frac{1}{2}u_0 & -v_1-\sqrt{\frac{3}{2}}u_1 \\ \sqrt{5}v_{-3} & -\sqrt{\frac{5}{2}}v_{-2} & v_{-1}+\sqrt{\frac{3}{2}}u_{-1} & -\frac{1}{2}v_0-\frac{3}{2}u_0 \end{pmatrix}. \tag{9}$$

The compact Lie algebras usp(4) and so(5) are isomorphic because of the low-dimensional Cartan isomorphism $C_2 \cong B_2$.

For $S = S(u, v)$ in usp(4), define the operator

$$\pi(S)=\sum_{\mu=-1}^{1} u_\mu \hat{L}_\mu + \sum_{\nu=-3}^{3} v_\nu \hat{O}_\nu, \tag{10}$$

where π determines a unitary representation of usp(4) since $\pi(S)^\dagger = \pi(S)$ and $\pi([S_1, S_2]) = [\pi(S_1), \pi(S_2)]$ for $S_1, S_2 \in$ usp(4). π extends to a representation of the complexification of usp(4) with $\pi(\mathcal{L}_\mu) = (-1)^\mu \hat{L}_{-\mu}$ and $\pi(\mathcal{O}_\nu) = (-1)^\nu \hat{O}_{-\nu}$.

The subalgebra su(2) of usp(4) consists of the matrices $S(u, v = 0)$, which is the four-dimensional $j = 3/2$ representation of the Lie algebra of the rotation group. The SU(2) subgroup of USp(4) consists of the 4×4 unitary matrices $R = \mathcal{D}^{(3/2)}(\alpha, \beta, \gamma)$, where α, β, γ are the Euler angles.

The rotation group SU(2) acts on the Lie algebra usp(4) by the adjoint transformation, $\mathrm{Ad}_R S = R \cdot S \cdot R^{-1}$. Since $\hat{L}^{(1)}$ and $\hat{O}^{(3)}$ are irreducible tensor operators and π is a representation, a rotated algebra element is represented by

$$\begin{aligned} \pi(\mathrm{Ad}_R S) &= \pi(R)\pi(S)\pi(R)^{-1} \\ &= \sum_{\mu,\mu'=-1}^{1} u_\mu \hat{L}_{\mu'} \mathcal{D}^{(1)}_{\mu'\mu}(\alpha,\beta,\gamma) + \sum_{\nu,\nu'=-3}^{3} v_\nu \hat{O}_{\nu'} \mathcal{D}^{(3)}_{\nu'\nu}(\alpha,\beta,\gamma) \\ &= \sum_{\mu'=-1}^{1} u_{\mu'} \hat{L}_{\mu'} + \sum_{\nu'=-3}^{3} v_{\nu'} \hat{O}_{\nu'}, \end{aligned} \tag{11}$$

where

$$u_{\mu'} = \sum_{\mu=-1}^{1} \mathcal{D}^{(1)}_{\mu'\mu}(\alpha,\beta,\gamma) u_\mu \qquad \text{and} \qquad v_{\nu'} = \sum_{\nu=-3}^{3} \mathcal{D}^{(3)}_{\nu'\nu}(\alpha,\beta,\gamma) v_\nu. \tag{12}$$

Thus, when $S = S(u, v)$ is rotated, u transforms as a vector and v transforms as an octupole tensor, i.e., $\mathrm{Ad}_R S(u, v) = S(\mathcal{D}^{(1)}(R)u, \mathcal{D}^{(3)}(R)v)$.

2. Density matrices

Given a state vector $|\Psi\rangle$ in the representation space $\mathcal{H}$, the expectations of the usp(4) operators are

$$\begin{aligned} l_\mu &= (-1)^\mu \langle\Psi|\hat{L}^{(1)}_{-\mu}|\Psi\rangle = \langle\Psi|\pi(\mathcal{L}_\mu)|\Psi\rangle \\ o_\nu &= (-1)^\nu \langle\Psi|\hat{O}^{(3)}_{-\nu}|\Psi\rangle = \langle\Psi|\pi(\mathcal{O}_\nu)|\Psi\rangle. \end{aligned} \tag{13}$$

These expectations are components of vector and octupole spherical tensors because $l_{-\mu} = (-1)^\mu (l_\mu)^*$ and $o_{-\nu} = (-1)^\nu (o_\nu)^*$, and l and o transform appropriately under a rotation.

When $|\Psi\rangle$ is rotated to $\pi(R)|\Psi\rangle$, the angular momentum expectation transforms from l_μ to $l_{\mu'}$,

$$\begin{aligned} l_{\mu'} &= (-1)^{\mu'}\langle\pi(R)\Psi|\hat{L}^{(1)}_{-\mu'}|\pi(R)\Psi\rangle \\ &= (-1)^{\mu'}\langle\Psi|\pi(R^{-1})\hat{L}^{(1)}_{-\mu'}\pi(R)|\Psi\rangle \\ &= (-1)^{\mu'}\sum_{\mu=-1}^{1}\langle\Psi|\hat{L}^{(1)}_{-\mu}|\Psi\rangle\mathcal{D}^{(1)}_{-\mu,-\mu'}(R^{-1}) \\ &= (-1)^{\mu'}\sum_{\mu=-1}^{1}(-1)^{-\mu}l_\mu\mathcal{D}^{(1)}_{-\mu,-\mu'}(R^{-1}) \\ &= \sum_{\mu=-1}^{1}\mathcal{D}^{(1)}_{\mu',\mu}(R)l_\mu. \end{aligned} \tag{14}$$

o transforms similarly as an octupole tensor.

Define the density matrix

$$\rho(l,o) = l\cdot\mathcal{L} + o\cdot\mathcal{O} = \sum_{\mu=-1}^{1}(-1)^{\mu}l_{-\mu}\mathcal{L}_\mu + \sum_{\nu=-3}^{3}(-1)^{\nu}o_{-\nu}\mathcal{O}_\nu, \tag{15}$$

or, explicitly,

$$\rho(l,o) = \begin{pmatrix} \frac{1}{2}o_0+\frac{3}{2}l_0 & -o_1-\sqrt{\frac{3}{2}}l_1 & \sqrt{\frac{5}{2}}o_2 & -\sqrt{5}o_3 \\ o_{-1}+\sqrt{\frac{3}{2}}l_{-1} & \frac{1}{2}l_0-\frac{3}{2}o_0 & \sqrt{3}o_1-\sqrt{2}l_1 & -\sqrt{\frac{5}{2}}o_2 \\ \sqrt{\frac{5}{2}}o_{-2} & \sqrt{2}l_{-1}-\sqrt{3}o_{-1} & \frac{3}{2}o_0-\frac{1}{2}l_0 & -o_1-\sqrt{\frac{3}{2}}l_1 \\ \sqrt{5}o_{-3} & -\sqrt{\frac{5}{2}}o_{-2} & o_{-1}+\sqrt{\frac{3}{2}}l_{-1} & -\frac{1}{2}o_0-\frac{3}{2}l_0 \end{pmatrix}. \tag{16}$$

The matrix ρ is an element of the dual space to the algebra. The pairing between a density $\rho\in\mathrm{usp}(4)^*$ and a algebra element $S\in\mathrm{usp}(4)$ is defined by

$$\langle\rho, S\rangle = \tfrac{1}{5}\,\mathrm{Tr}(\rho S), \tag{17}$$

and this real number equals the expectation of the operator $\pi(S)$ in the state $|\Psi\rangle$,

$$\begin{aligned} \langle\rho, S\rangle &= \sum_{\mu=-1}^{1}(-1)^{\mu}l_\mu u_{-\mu} + \sum_{\nu=-2}^{2}(-1)^{\nu}o_\nu v_{-\nu} \\ &= \langle\Psi|\pi(S)|\Psi\rangle. \end{aligned} \tag{18}$$

When π is an irreducible representation of usp(4) and $|\Psi\rangle$ is a highest weight vector, the density is a diagonal matrix,

$$\varrho = \frac{5}{2}\begin{pmatrix} m_1+m_2 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & -m_2 & 0 \\ 0 & 0 & 0 & -m_1-m_2 \end{pmatrix}, \tag{19}$$

where m_1 and m_2 are the nonnegative integral weights that label π,

$$\begin{aligned} m_1 &= \langle\varrho, E_{11}-E_{22}+E_{33}-E_{44}\rangle \\ m_2 &= \langle\varrho, E_{22}-E_{33}\rangle. \end{aligned} \tag{20}$$

2.1. Coadjoint orbits

The unitary symplectic group acts on its Lie algebra by the adjoint transformation, $\mathrm{A}d_g S = g \cdot S \cdot g^{-1}$ for $S \in \mathrm{usp}(4)$ and $g \in \mathrm{USp}(4)$. This induces the coadjoint action of USp(4) on the dual space $\mathrm{usp}(4)^*$, $\mathrm{A}d^*_g \rho = g \cdot \rho \cdot g^{-1}$ for $\rho \in \mathrm{usp}(4)^*$ and $g \in \mathrm{USp}(4)$. The coadjoint and adjoint actions are related by the pairing, $\langle \mathrm{A}d^*_g \rho, S\rangle = \langle \rho, \mathrm{A}d_{g^{-1}} S\rangle$. When ρ is the density corresponding to the state $|\Psi\rangle$, $\mathrm{A}d^*_g \rho$ is the density corresponding to the state $\pi(g)|\Psi\rangle$. The coadjoint orbit containing the point ρ consists of ρ and all the transformed densities $\mathrm{A}d^*_g \rho$ as g ranges over the group USp(4). The mean field approximation limits the model densities to one coadjoint orbit.

Each coadjoint orbit contains a diagonal matrix ϱ; the proof is provided in the appendix. When the order of the eigenvalues is fixed, the coadjoint orbits are enumerated uniquely by

$$\mathcal{O}_\varrho = \{\mathrm{A}d^*_g \varrho | g \in \mathrm{USp}(4)\}. \tag{21}$$

where ϱ is given by equation (19) with m_1, m_2 nonnegative real numbers. The orbits $\mathcal{O}_\varrho$ do not intersect when m_1, m_2 are restricted to nonnegative real numbers. The integral orbits are those with m_1, m_2 nonnegative integers. The geometric quantization method or the Borel–Weil–Bott theorem associates naturally an irreducible unitary representation of USp(4) with each integral coadjoint orbit.

Each coadjoint orbit $\mathcal{O}_\varrho$ is diffeomorphic to a homogeneous space that equals the group USp(4) modulo the isotropy subgroup H_ϱ at ϱ. The isotropy subgroup consists of the USp(4) group elements g that commute with ϱ. There are three possibilities depending on m_1, m_2,

$$H_\varrho = \begin{cases} U(1) \times U(1), & m_1, m_2 > 0 \\ U(2), & m_1 = 0 \quad \text{or} \quad m_2 = 0 \\ USp(4), & m_1 = m_2 = 0. \end{cases} \tag{22}$$

The dimension of a homogeneous space, $\mathrm{USp}(4)/H_\varrho$, is the difference between the dimension of USp(4) and the dimension of the isotropy subgroup,

$$\dim \mathcal{O}_\varrho = \begin{cases} 8, & m_1, m_2 > 0 \\ 6, & m_1 = 0 \quad \text{or} \quad m_2 = 0 \\ 0, & m_1 = m_2 = 0. \end{cases} \tag{23}$$

The Casimir invariants,

$$\mathcal{C}^{(n)}(\rho) = \tfrac{1}{5} \operatorname{tr}(\rho^n), \tag{24}$$

are real-valued constant functions on each coadjoint orbit, $\mathcal{C}^{(n)}(\mathrm{A}d^*_g \rho) = \mathcal{C}^{(n)}(\rho)$ for $g \in \mathrm{USp}(4)$ and $\rho \in \mathrm{usp}(4)^*$. The value of a Casimir may be computed most easily at the orbit representative ϱ. When n is odd, the Casimir equals zero. For $n = 2, 4$ the Casimirs equal

$$\begin{aligned} \mathcal{C}^{(2)}(\varrho) &= \tfrac{5}{2}\left(2m_2^2 + 2m_1 m_2 + m_1^2\right) \\ \mathcal{C}^{(4)}(\varrho) &= \tfrac{125}{8}\left(2m_2^4 + 4m_1 m_2^3 + 6m_1^2 m_2^2 + 4m_1^3 m_2 + m_1^4\right). \end{aligned} \tag{25}$$

The Casimirs of higher degree $n \geqslant 6$ are not functionally independent of the quadratic and quartic invariants.

After expressing the density matrix explicitly as $\rho = \rho(l, o)$, the two independent Casimirs are calculated to be

$$\mathcal{C}^{(2)}(\rho) = l \cdot l + o \cdot o \tag{26}$$

$$\mathcal{C}^{(4)}(\rho) = \frac{1}{4}(\mathcal{C}^{(2)}(\rho))^2 + 21[o\times o]^{(2)}\cdot[o\times o]^{(2)} + \mathcal{C}^{(2)}(\rho) l\cdot l$$
$$- (l\cdot l)^2 + 3\sqrt{10}\{[[l\times l]^{(2)}\times l]^{(3)}\cdot o + \text{c.c.}\}$$
$$- 3\sqrt{\frac{7}{2}}\{[[l\times l]^{(2)}\cdot[o\times o]^{(2)} + \text{c.c.}\} + 21\{[[l\times o]^{(2)}\cdot[o\times o]^{(2)} + \text{c.c.}\} \qquad (27)$$

For a compact Lie group, a coadjoint orbit is identical to a common level surface of its Casimir functions. The characterization of a coadjoint orbit as a level surface is useful for computations. This level surface in the dual space $\text{usp}(4)^*$ is an algebraic surface defined by two polynomial Casimir equations in the complex variables l_μ and o_ν that make up a density matrix.

3. Symplectic geometry

This section defines the symplectic geometry on each coadjoint orbit $\mathcal{O}_\varrho$ and uses this structure to associate a Hamiltonian vector field with each smooth real-valued function on $\mathcal{O}_\varrho$. The symplectic form ω_ρ at any point ρ of the orbit is a closed, nondegenerate 2-form that is equivalent to a Poisson bracket. This form determines the dynamics of usp(4) densities from a given energy function.

Suppose ρ is any point on a coadjoint orbit $\mathcal{O}_\varrho$. Every Lie algebra element S defines a vector $\bar{S}$ at each point ρ that is tangent to the surface $\mathcal{O}_\varrho$ in the dual space: $\bar{S}$ denotes the tangent to the curve $\epsilon \mapsto \exp(-\mathrm{i}\epsilon S)\cdot\rho\cdot\exp(\mathrm{i}\epsilon S)$. When S is in the annihilator $\mathcal{A}_\rho$ at ρ, $[S,\rho]=0$, the vector $\bar{S}$ vanishes. Two tangent vectors $\bar{S}$ and $\bar{T}$ are equal when $S-T\in\mathcal{A}_\rho$. Thus the tangent space to the orbit $\mathcal{O}_\varrho$ at ρ may be identified with the coset space $\text{usp}(4)/\mathcal{A}_\rho$.

For $S, T\in\text{usp}(4)$, define the symplectic form at ρ by

$$\omega_\rho(\bar{S},\bar{T}) = -\mathrm{i}\langle\rho,[S,T]\rangle. \qquad (28)$$

This form is well defined since, if $S-S'\in\mathcal{A}_\rho$ and $T-T'\in\mathcal{A}_\rho$, then $\langle\rho,[S,T]\rangle = \langle\rho,[S',T']\rangle$. The form is evidently antisymmetric. Moreover, because usp(4) is a simple Lie algebra with a nondegenerate Killing form, ω_ρ is likewise nondegenerate, i.e., $\omega_\rho(\bar{S},\bar{T})=0$ for all tangents $\bar{T}$ at ρ implies that $\bar{S}$ vanishes. As a consequence of the Jacobi identity, this form is also closed. These various facts about ω_ρ and the symplectic geometry on any coadjoint orbit $\mathcal{O}_\varrho$ are well known [35].

Given any smooth real-valued function f on the orbit $\mathcal{O}_\varrho$, there exists a vector field $\bar{S}_f$ on the orbit surface such that

$$\omega_\rho(\bar{S}_f,\bar{T}) = \mathrm{d}f(\bar{T})$$
$$= \frac{\mathrm{d}}{\mathrm{d}\epsilon} f(\exp(-\mathrm{i}\epsilon T)\cdot\rho\cdot\exp(\mathrm{i}\epsilon T))|_{\epsilon=0} \qquad (29)$$

for all tangents $\bar{T}$ to the orbit surface at ρ. There is a unique solution, $\bar{S}_f$, to this equation because the symplectic form is nondegenerate. Naturally the Lie algebra element S_f is not unique as any element of the annihilator at ρ may be added to S_f. $\bar{S}_f$ is called the Hamiltonian vector field associated with f even when f is not the energy. When $f=\mathcal{E}$ is the energy function, the Hamiltonian vector field is the mean field Hamiltonian, $\bar{h}=\bar{S}_\mathcal{E}$,

$$\omega_\rho(\bar{h[\rho]},\bar{T}) = \mathrm{d}\mathcal{E}(\bar{T}), \qquad (30)$$

for all $T\in\text{usp}(4)$.

For each $S\in\text{usp}(4)$, consider the elementary real-valued function on the dual space, $\lambda(S)(\rho)=\langle\rho,S\rangle$. The value of the function $\lambda(S)$ at the density ρ equals the expectation of the observable S with respect to any state whose density equals ρ. It is proven easily that the

Table 2. Hamiltonian vector fields $S(u, v) \in \mathrm{usp}(4)$ corresponding to some smooth functions f of the density.

Function f	u	v
$l \cdot l$	$2l$	0
$o \cdot o$	0	$2o$
$(l \cdot l)^2$	$4(l \cdot l)l$	0
$(o \cdot o)^2$	0	$4(o \cdot o)o$
$[o \times o]^{(2)} \cdot [o \times o]^{(2)}$	0	$-4\sqrt{\frac{5}{7}}[[o \times o]^{(2)} \times o]^{(3)}$
$[l \times [o \times o]^{(2)}]^{(1)} \cdot l$	$2[l \times [o \times o]^{(2)}]^{(1)}$	$2\sqrt{\frac{5}{7}}[[l \times l]^{(2)} \times o]^{(3)}$
$\mathcal{C}_r$	0	0

Hamiltonian vector field associated with the function $\lambda(S)$ is $\bar{S}$. In particular, the Hamiltonian vector fields associated with the 'coordinate' functions l_μ and o_ν are $\bar{\mathcal{L}}_\mu$ and $\bar{\mathcal{O}}_\nu$, respectively, (cf equation (13)).

The Hamiltonian vector field associated with a function f on $\mathcal{O}_\varrho$ that is itself a function of the 'coordinate' functions may be computed using the properties of the differential. For example, when $f = l \cdot l$, the differential is

$$\mathrm{d}f = \sum_\mu (-1)^\mu (\mathrm{d}l_{-\mu} l_\mu + l_{-\mu} \mathrm{d}l_\mu) = 2 \sum_\mu (-1)^\mu l_{-\mu}\, \mathrm{d}l_\mu. \tag{31}$$

The Hamiltonian vector field is

$$\bar{S}_{l\cdot l} = 2l \cdot \bar{\mathcal{L}}. \tag{32}$$

The Hamiltonian vector fields associated with various smooth functions are provided in table 2.

The Casimirs are constant functions on each coadjoint orbit and their differentials must, therefore, vanish. To verify this, note that the Hamiltonian vector field associated with the quadratic Casimir, equation (26), is $2(l \cdot \mathcal{L} + o \cdot \mathcal{O}) = 2\rho \in \mathcal{A}_\rho$.

3.1. Dynamics on $\mathcal{O}_\varrho$

A geometrical condition determines the time evolution of a usp(4) density matrix: a solution $\rho(t)$ must be an integral curve of the mean field Hamiltonian $\overline{h}[\rho]$ or

$$\mathrm{i}\frac{\mathrm{d}\rho}{\mathrm{d}t} = [h[\rho], \rho]. \tag{33}$$

Equation (33) is a finite-dimensional Lax equation [36, 37]. It is formally the same as the time-dependent Hartree–Fock equation [38].

Dynamics may be expressed equivalently using the Poisson bracket. The Poisson bracket on $\mathcal{O}_\varrho$ is defined from the symplectic form. The bracket of two smooth real-valued functions f, g on $\mathcal{O}_\varrho$ is

$$\{f, g\}(\rho) \equiv \omega_\rho(\overline{S}_f[\rho], \overline{S}_g[\rho]). \tag{34}$$

When f is any smooth function on a coadjoint orbit, its time rate of change along a solution curve is

$$\dot{f} = \{f, \mathcal{E}\}. \tag{35}$$

For example, when $f = \lambda(S)$, the time rate of change of the observable corresponding to S along a solution curve is

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda(S) = \langle \dot{\rho}, S\rangle = \omega_\rho(\overline{S}, \overline{h}[\rho]) = \{\lambda(S), \mathcal{E}\}. \tag{36}$$

The last line can be written alternatively as the derivative of $\mathcal{E}$ in the direction $\overline{S}$, $\{\lambda(S), \mathcal{E}\} = -\mathrm{d}\mathcal{E}(\overline{S})$.

3.2. Rotation group SU(2)

The rotation group SU(2) is an embedded Lie subgroup of USp(4) corresponding to the $\mathcal{D}^{(3/2)}$ irreducible representation. When $R \in \mathrm{SU}(2)$ acts on a density $\rho(l, o)$, it rotates l and o as vector and octupole tensors. Each orbit of the rotation group contains a density with the angular momentum vector aligned with the 3-axis, $l_1 = l_{-1} = 0$. A density with the angular momentum pointing along the 3-axis may be rotated about the 3-axis without changing the angular momentum vector. Each such orbit of the subgroup of rotations about the 3-axis contains an octupole expectation o_1 that is real.

Let $\mathcal{M}_\varrho$ denote the following surface of all densities contained in the coadjoint orbit $\mathcal{O}_\varrho$:

$$\mathcal{M}_\varrho = \{\tilde{\rho} = \rho(l, o) \in \mathcal{O}_\varrho | l_1 = l_{-1} = 0, o_{-1} = -o_1\}. \tag{37}$$

Each orbit of SU(2) in $\mathcal{O}_\varrho$ contains a density in the submanifold $\mathcal{M}_\varrho$. The densities $\tilde{\rho}$ in $\mathcal{M}_\varrho$ represent the density in the intrinsic frame. The space of intrinsic frame densities $\mathcal{M}_\varrho$ is five dimensional when $m_1, m_2 > 0$.

When the energy function is a rotational scalar, transformation of the dynamical system to the intrinsic frame simplifies the analysis. Let $R(t) \in \mathrm{SU}(2)$ be a smooth time-dependent rotation that transforms a solution curve of the dynamical system (33) into the submanifold of intrinsic densities. Define the time-dependent matrix

$$\Omega = -\mathrm{i}\frac{\mathrm{d}R}{\mathrm{d}t}R^{-1} = \sum_\mu \omega_\mu \mathcal{L}_\mu. \tag{38}$$

The pseudo-vector $\vec{\omega}$ corresponding to the matrix Ω is the angular velocity. Let $\tilde{\rho}(t) = R \cdot \rho \cdot R^{-1} \in \mathcal{M}_\varrho$ denote the density in the intrinsic frame. The Hamiltonian dynamical system on the coadjoint orbit, equation (33), is equivalent to the following dynamical equation on $\mathcal{M}_\varrho$:

$$\mathrm{i}\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} = [h_\Omega[\tilde{\rho}], \tilde{\rho}] \tag{39}$$

where $h_\Omega[\tilde{\rho}] = Rh[\rho]R^{-1} - \Omega = h_{[}\tilde{\rho}] - \Omega$ is the usp(4) Routhian. When the energy function is a rotational scalar, the Hamiltonian vector field is invariant under rotations, $h[\tilde{\rho}] = R \cdot h[\rho] \cdot R^{-1}$.

3.3. Range of the angular momentum

The object of this section is to prove that the maximum value of the angular momentum on an usp(4) integral coadjoint orbit coincides with its maximum value in the corresponding irreducible highest weight representation. The maximum value of the squared length, $l \cdot l$, of the angular momentum is attained on any coadjoint orbit because every usp(4) orbit is compact and $l \cdot l$ is continuous. Such a maximum is a critical point of the smooth function $l \cdot l$, or the corresponding Hamiltonian vector field, $X_{l \cdot l} = 2l \cdot \mathcal{L}$, vanishes at a critical point. Since $l \cdot l$ is constant on the orbits of SU(2) it is sufficient to identify the critical points $\tilde{\rho}$ on $\mathcal{M}_\varrho$:

$$\begin{aligned} 0 &= [X_{l \cdot l}, \tilde{\rho}] = [2l_0\mathcal{L}_0, l_0\mathcal{L}_0 + o \cdot \mathcal{O}] = 2l_0[\mathcal{L}_0, o \cdot \mathcal{O}] \\ &= 2l_0 \sum_\nu (-1)^\nu o_{-\nu}(-\nu)\mathcal{O}_\nu. \end{aligned} \tag{40}$$

Table 3. The critical points of the length of the angular momentum vector in $\mathcal{M}_\varrho$.

$2l_0$	$2o_0$
$3m_1 + 4m_2$	$m_1 - 2m_2$
$3m_1 + 2m_2$	$m_1 + 4m_2$
$m_1 + 4m_2$	$-3m_1 - 2m_2$
$-m_1 + 2m_2$	$3m_1 + 4m_2$
$m_1 - 2m_2$	$-3m_1 - 4m_2$
$-m_1 - 4m_2$	$3m_1 + 2m_2$
$-3m_1 - 2m_2$	$-m_1 - 4m_2$
$-3m_1 - 4m_2$	$-m_1 + 2m_2$

Therefore, the product $l_0 \nu o_\nu = 0$ for all ν at a critical point of $l \cdot l$. There are two potential solutions, one that is not rotating, $l = 0$, and the other that is a rotating 'pear-like' system, $o_\nu = 0$ when $\nu \neq 0$.

To complete the derivation, it is necessary to impose the condition that $\tilde{\rho}$ is a point on the surface $\mathcal{M}_\varrho$. This condition is ensured when $\tilde{\rho}$ satisfies the two Casimir equations. When $l_0 \neq 0$, table 3 enumerates the critical points of $l \cdot l$ on the surface $\mathcal{M}_\varrho$. The maximum angular momentum solution is

$$l_0 = \tfrac{1}{2}(3m_1 + 4m_2) \qquad o_0 = \tfrac{1}{2}(m_1 - 2m_2). \tag{41}$$

This maximal angular momentum value is the same as that found for irreducible highest weight representations of usp(4) with weights $[m_1, m_2]$.

3.4. Energy function

The energy function $\mathcal{E}(\rho)$ is a real-valued function defined on the dual space usp(4)*. This function is assumed to be invariant under rotations, $\mathcal{E}(R \cdot \rho \cdot R^{-1}) = \mathcal{E}(\rho)$ for all $R \in \mathrm{SU}(2)$. As a consequence, the angular momentum vector is constant along each solution curve, e.g., for the null component of l,

$$\dot{l}_0 = \{l_0, \mathcal{E}\}(\rho) = -\mathrm{d}\mathcal{E}(\overline{T})(\rho) = -\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{E}(R(-\theta) \cdot \rho \cdot R(\theta))|_{\theta=0} = 0, \tag{42}$$

where $l_0 = \langle \rho, T \rangle$ is the null component of the angular momentum vector when $T = \mathrm{diag}(3/2, 1/2, -1/2, -3/2)$ and $R(\theta) = \exp(\mathrm{i}\theta T) = \mathcal{D}^{(3/2)}(\alpha = 0, \beta = 0, \gamma = \theta)$ is a rotation in the 1–2 plane through an angle θ.

A simple model energy function is

$$\mathcal{E}_0(\rho) = A_1(l \cdot l) + A_2(l \cdot l)^2 + A_3(o \cdot o) + A_4(o \cdot o)^2 + A_5(l \cdot l)(o \cdot o), \tag{43}$$

where A_k are the real constants. When the $(o \cdot o)$ terms are replaced by the difference between the quadratic usp(4) Casimir and $l \cdot l$, the resulting energy function is a pure function of $l \cdot l$. A rotating equilibrium density $\tilde{\rho}$, by definition, commutes with the Hamiltonian vector field $h_\Omega[\tilde{\rho}]$ corresponding to $\mathcal{E}_0$. For this rather trivial energy function $\mathcal{E}_0(\rho)$, every density $\tilde{\rho}$ that lies in $\mathcal{M}_\varrho$ can be shown to be an equilibrium solution.

A more interesting model energy function is

$$\mathcal{E}(\rho) = A(l \cdot l + \chi[o \times o]^{(2)} \cdot [o \times o]^{(2)}), \tag{44}$$

where A and χ are the real constants. Unlike $\mathcal{E}_0(\rho)$ this energy cannot be written as a pure function of the angular momentum. Table 2 gives the Hamiltonian vector field associated with $\mathcal{E}(\rho)$. A density $\tilde{\rho} \in \mathcal{M}_\varrho$ is a rotating equilibrium solution when it commutes with $h_\Omega[\tilde{\rho}]$.

The angular velocity Ω is determined by requiring that the commutator matrix $[\tilde{\rho}, h_\Omega[\tilde{\rho}]]$ lies in $\mathcal{M}_\varrho$. This commutator matrix vanishes when $o_1 = o_2 = 0$. As a consequence, the angular velocity vector is aligned with the 3-axis, $\omega_1 = \omega_{-1} = 0$, with magnitude

$$\frac{\omega_0}{2A} = l_0 + \frac{1}{21}\chi(5l_0 + 3o_0)\big(2o_0^2 - 5|o_3|^2\big). \tag{45}$$

The density $\tilde{\rho}$ also satisfies the two Casimir equations,

$$\begin{aligned}
&\mathcal{C}^{(2)}(\varrho) = l_0^2 + o_0^2 + 2|o_3|^2\\
&\mathcal{C}^{(4)}(\varrho) = \tfrac{1}{20}\big(41l_0^4 + 48o_0l_0^3 + 54o_0^2l_0^2 - 48o_0^3l_0 + 41o_0^4\big) + |o_3|^2\big(9l_0^2 + 6l_0o_0 + o_0^2 + 10|o_3|^2\big).
\end{aligned} \tag{46}$$

There are four distinct solutions to the two Casimir equations:

$$(1)\ \begin{cases} o_0 = \frac{1}{3}(l_0 - 5m_2) \\ |o_3|^2 = \frac{5}{36}(3m_1 + 2m_2 + 2l_0)(3m_1 + 4m_2 - 2l_0) \end{cases} \tag{47}$$

$$(2)\ \begin{cases} o_0 = \frac{1}{3}(l_0 + 5m_2) \\ |o_3|^2 = \frac{5}{36}(3m_1 + 2m_2 - 2l_0)(3m_1 + 4m_2 + 2l_0) \end{cases} \tag{48}$$

$$(3)\ \begin{cases} o_0 = \frac{1}{3}(l_0 - 5(m_1 + m_2)) \\ |o_3|^2 = \frac{5}{36}(2m_2 - m_1 + 2l_0)(m_1 + 4m_2 - 2l_0) \end{cases} \tag{49}$$

$$(4)\ \begin{cases} o_0 = \frac{1}{3}(l_0 + 5(m_1 + m_2)) \\ |o_3|^2 = \frac{5}{36}(2m_2 - m_1 - 2l_0)(m_1 + 4m_2 + 2l_0). \end{cases} \tag{50}$$

Each solution is limited to a maximum value of l_0 since $|o_3|^2$ is nonnegative. For example, the maximum value of the angular momentum for solution (1) is $l_0 = (3m_1 + 4m_2)/2$ (cf equation (41)). The solutions are axially asymmetric because o_3 is nonzero with the exception of a band's density at its maximum angular momentum. For the $(m_1, m_2) = (2, 2)$ coadjoint orbit, figure 2 shows the octupole components o_0 and $|o_3|$ of solution (1) versus the angular momentum l_0.

The energies of these rotating equilibrium solutions are

$$\mathcal{E}(\tilde{\rho})/A = l_0^2 + \frac{\chi}{21}\big(2o_0^2 - 5|o_3|^2\big)^2. \tag{51}$$

For band (1) solutions, the energy specializes to

$$\mathcal{E}(l_0)/A = l_0^2 + \frac{\chi}{336}(4l_0(3l_0 - 5m_2) - 25m_1(2m_2 + m_1))^2. \tag{52}$$

The angular velocity, equation (45), for band (1) equals the derivative of this energy with respect to the angular momentum, $\omega_0 = \mathrm{d}\mathcal{E}/\mathrm{d}l_0$. When $(m_1, m_2) = (2, 2)$, figure 3 shows the energy of solution (1) against the squared length of the angular momentum. For $\chi = 0$, the energy is a straight line in this graph indicating that the band has a constant moment of inertia. At the top of the band at $l_0 = 7$ the energy is independent of χ. The moment of inertia is generally smaller (greater) for more negative (positive) values of χ than the trivial $\chi = 0$ case. For $\chi \approx 0.10$ and $2 < l_0 < 5$, the energies of the states differ slightly which indicates a large moment of inertia in this domain.

When $m_1 = 0$, the integral orbits $(0, m_2)$ correspond to the symmetric representations. Figure 4 shows the energy of band (1) for various values of χ. If $m_1 = 0$ the term

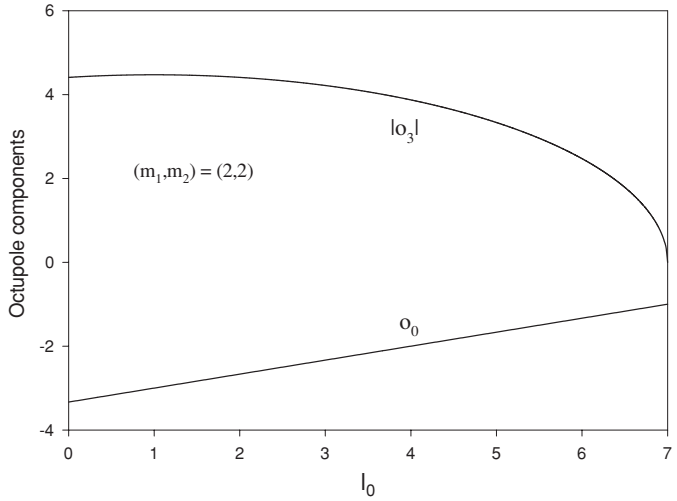

Figure 2. Octupole components o_0 and $|o_3|$ of equation (47) versus the angular momentum l_0 for $(m_1, m_2) = (2, 2)$.

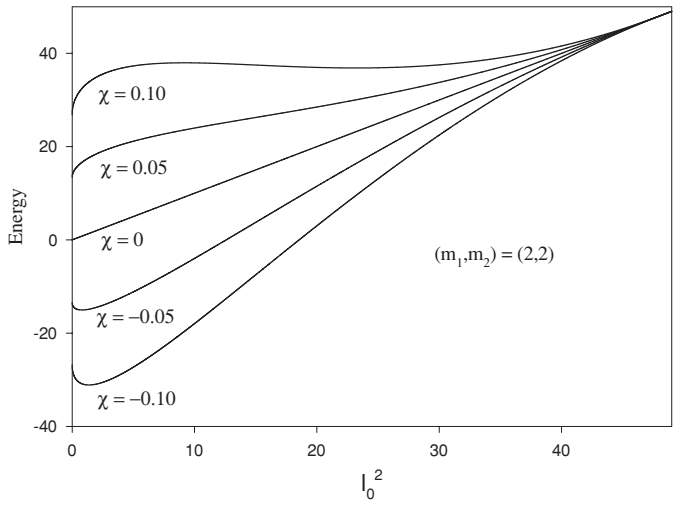

Figure 3. When $(m_1, m_2) = (2, 2)$, the energy of band (1) densities, equation (52), in units of A, is plotted versus the square of the angular momentum l_0^2 for various values of χ.

$[o \times o]^{(2)} \cdot [o \times o]^{(2)}$ is zero when $l_0 = 5m_2/3$ and the energy is independent of χ for this angular momentum value.

4. Discussion

The algebraic mean field theory was used in this paper to derive analytical expressions for the energy, equation (51), and octupole deformations, equations (47)–(50), in the usp(4) ≅ so(5) model. An analytical formula for the energy, such as equation (52) for band (1) solutions, provides an immediate understanding of the dependence of the energy on the angular

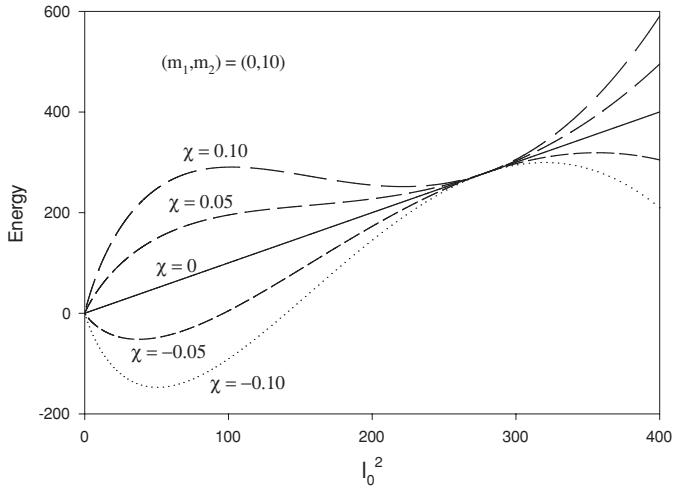

Figure 4. When $(m_1, m_2) = (0, 10)$, the energy of band (1) densities, equation (52), in units of A is plotted versus the square of the angular momentum l_0^2 for various values of χ.

momentum and the orbit data (m_1, m_2). Mathematical computations in the algebraic mean field theory require matrix operations using a faithful representation of the algebra and mean field solutions are obtained by solving a system of polynomial equations. For example, the usp(4) mean field theory uses 4×4 complex matrices, and the usp(4) equilibrium equations simplified to the polynomial equation system (46). In contrast, the usp(4) representation theory uses a vector space whose dimension, depending on the particular application, can be unmanageably large. The determinations of the eigenvalue spectrum and octupole deformations in irreducible representations require numerical diagonalization of the Hamiltonian, and, except for special irreps, analytical expressions for the energy and octupole expectations cannot be derived. Thus, mean field theory is simpler computationally than representation theory.

The coadjoint orbit in mean field theory is not restricted to an integral orbit which corresponds to a highest weight representation. The introduction reviewed an interesting example of weak dynamical symmetry for su(3) in the context of a quantum phase transition in the interacting boson model. When the Hamiltonian is an interpolation, $H_\alpha = \alpha H_1 + (1 - \alpha)H_2$, of two pure dynamical symmetry Hamiltonians, H_1 and H_2, its eigenstates tend to fall into one of the two dynamical symmetry phases associated with either H_1 or H_2 [39–41]. Although the eigenstates in a phase are typically admixtures of vectors from many irreducible representation subspaces and dynamical symmetry is strongly broken, the densities in a symmetry phase are points on a single nonintegral coadjoint orbit and weak dynamical symmetry is nonetheless present. Note that mean field theory for nonintegral orbits is no less tractable than that for integral orbits. In particular, formulae such as equation (52) are valid when m_1 and m_2 are any nonnegative real numbers.

For finite-dimensional simple compact Lie algebras, the level surface of the Casimirs is a coadjoint orbit of a Lie group and, hence, a symplectic manifold. Given an energy function on the dual space, its restriction to a coadjoint orbit defines a Hamiltonian vector field on the coadjoint orbit. This Hamiltonian may be expressed as a function of the density with values in the Lie algebra. It is called a 'mean field' Hamiltonian because of the dependence on the density which is defined by the mean expectations of the algebra operators. An energy function exists on the dual space whose minimum is the density of the exact ground state

energy of the system [42]; this aspect of algebraic mean field theory involves a generalization of density functional theory and the celebrated Hohenberg–Kohn theorem [43]. The proof of the existence of the energy function is not constructive and the explicit determination of the energy is an open problem.

The methods and results of the theory of Hamiltonian dynamical systems may be applied to the new class of physical problems defined by mean field theory. One interesting problem is to discover physically reasonable energy functions that define integrable Hamiltonian systems. Note that, although the Toda lattice is integrable [44], it is not relevant to mean field descriptions of many-body systems because the energy function must be a rotational scalar.

An integral orbit corresponds to an irrep of the Lie group. One way to determine various properties of an irrep is to infer them directly from the geometry of its integral coadjoint orbit and the range of certain real-valued smooth functions on the orbit. This method is advocated persuasively by Kirillov [35]. Some rigorous general theorems are known about the correspondence that can be applied to specific groups, e.g., Kirillov's character formula [45]. But applied group representation theoreticians have not explored systematically the orbit method to derive irrep properties. An important open problem is to discover an efficient technique to evaluate multiplicities of subgroup irreps from the integral coadjoint orbit data [46, 47]. When specialized to an integral orbit, the mean field method is essentially the determination of irrep properties directly from the orbit space.

Suppose a state vector Ψ is a linear combination of normalized highest weight vectors $|(m_1, m_2)HW\rangle$ contained in a reducible usp(4) unitary representation π: $\Psi = \sum A(m_1, m_2)|(m_1, m_2)HW\rangle$, where $A(m_1, m_2)$ are, in general, complex coefficients with $\sum |A(m_1, m_2)|^2 = 1$. The density matrix ϱ corresponding to the vector Ψ satisfies the defining condition, $\langle \varrho, S\rangle = \langle\Psi|\pi(S)|\Psi\rangle = \sum |C(m_1, m_2)|^2 \langle (m_1, m_2)HW|\pi(S)|(m_1, m_2)HW\rangle$ for all $S \in$ usp(4). The matrix elements in this sum are zero unless S is in the usp(4) Cartan subalgebra, and, therefore, the density is a real diagonal matrix, $\varrho = \mathrm{diag}(a_1 + a_2, a_2, -a_2, -a_1 - a_2)$, where a_1, a_2 are nonnegative real numbers, $a_k = \sum m_k |A(m_1, m_2)|^2$.

The states in the set $\mathcal{C}(\Psi) = \{\pi(g)\Psi | g \in \mathrm{USp}(4)\}$ are called coherent states [48–51]. If ϱ is the density corresponding to Ψ, then $\mathrm{Ad}^*_g\varrho$ is the density corresponding to the coherent state $\pi(g)\Psi$:

$$\begin{aligned} &\mathcal{C}(\Psi) \to \mathcal{O}_\varrho \\ &\pi(g)\Psi \mapsto \mathrm{Ad}^*_g\varrho. \end{aligned} \tag{53}$$

If Ψ is a highest weight vector, then the mapping, equation (53), is a diffeomorphism and $\mathcal{O}_\varrho$ is an integral coadjoint orbit. In this favourable case the space of coherent states is a symplectic manifold, and the properties of π may be determined by quantizing its coherent states, a method that is closely related to the geometric quantization construction for an integral coadjoint orbit [51]. When Ψ is a linear combination of highest weight vectors, equation (53) is a many-to-one mapping, and the space of coherent states no longer inherits the symplectic geometry of a coadjoint orbit.

Appendix

Every coadjoint orbit of USp(4) contains a diagonal matrix. The object of this appendix is to prove this assertion using elementary linear algebra. In all likelihood, the proof was published already elsewhere, but, unfortunately, I do not know a reference for it. Since usp(4) is simple, the dual space and the algebra may be identified and the coadjoint and adjoint actions coincide. An equivalent proposition is that every adjoint orbit of USp(4) contains a diagonal matrix.

Suppose S is a Lie algebra matrix in usp(4). Since S is Hermitian, there exists a unitary matrix $g \in \mathrm{U}(4)$ that diagonalizes S, $\mathrm{Ad}_{g^{-1}} S = g^{-1} \cdot S \cdot g = \mathrm{diag}(a, b, c, d)$, and the eigenvalues a, b, c, d are real. The columns of g are four linearly independent eigenvectors of S. The problem is to show that a diagonalizing matrix g may be chosen from the Usp(4) subgroup of U(4).

Lemma 1. *If $v \in C^4$ is an eigenvector of S belonging to the eigenvalue a, then Jv^* is another linearly independent eigenvector of S belonging to the eigenvalue $-a$. The eigenvectors may be chosen orthonormal, $v^\dagger v = (Jv^*)^\dagger (Jv^*) = 1$, $v^\dagger Jv^* = 0$.*

If $Sv = av$, then $S(Jv^*) = S^\dagger Jv^* = -J(S^* v^*) = -J(av^*) = -a(Jv^*)$. Because S is Hermitian, when $a \neq 0$ v and Jv^* are necessarily orthogonal, $v^\dagger Jv^* = 0$, and the two eigenvectors are linearly independent. Even when $a = 0$ the two eigenvectors are linearly independent. Suppose, in contrast, they are linearly dependent so that $Jv^* = \lambda v$, where $\lambda \neq 0$. In this case, $J^2 v^* = -v^* = \lambda Jv = |\lambda|^2 v^*$, which implies a contradiction, namely, $|\lambda|^2 = -1$. When $a = 0$, the two independent vectors v and Jv^* need not be orthogonal. But when they are not orthogonal, linear combinations may be chosen that are orthogonal, namely,

$$\tilde{v} = |v^\dagger Jv^*| v + \mathrm{i}(v^\dagger Jv^*)^* Jv^* \tag{A.1}$$

$$J\tilde{v}^* = |v^\dagger Jv^*| Jv^* + \mathrm{i}(v^\dagger Jv^*) v. \tag{A.2}$$

If the eigenvalues of S are $a, b, -a, -b$ with $a, b \geqslant 0$ and the corresponding eigenvectors are v, w, Jv^*, Jw^*, then these vectors may be chosen to be an orthonormal basis of C^4. When $a \neq b$ the eigenvectors of the Hermitian matrix S must be orthogonal. When $a = b$, Gram–Schmidt orthogonalization constructs an orthogonal basis in the two-dimensional degenerate eigenspace. If these four orthogonal vectors are normalized, define the 4×4 matrix g whose columns in order from left to right are v, w, Jw^*, Jv^*. The matrix g is unitary because the columns are orthonormal. One verifies by direct calculation that g is symplectic: ${}^T g \cdot J \cdot g = J$.

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